31: Algebraic and Metric Properties of Random Geometric Graphs and Complexes

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Requested positions: Doctoral students: 1 at the University of Osnabrück
Possible connections to projects: 4, 7, 10, 13, 28, 32, 34

Gilbert Graph and Random Simplicial Complexes

- **Poisson point process** η_t: in $\mathbb{R}^d$ with intensity measure $\Theta(A) = t\lambda_d(A) > 0$
- **Gilbert graph** on η_t: vertices $\mathcal{F}_0 = \eta_t$
  edges $\mathcal{F}_1 = \{\{v_0, v_1\} \subset \eta_t : B(v_0, \frac{\delta_t}{2}) \cap B(v_1, \frac{\delta_t}{2}) \neq \emptyset\}$
- **Vietoris-Rips complex** $\mathcal{V}(\eta_t, \delta_t)$: clique complex of the Gilbert graph
  k-simplices $\mathcal{F}_k = \left\{\{v_0, \dots, v_k\} \subset \eta_t : B(v_i, \frac{\delta_t}{2}) \cap B(v_j, \frac{\delta_t}{2}) \neq \emptyset\right\}$
- **Čech complex** $\mathcal{C}(\eta_t, \delta_t)$: nerve of the Boolean model
  k-simplices $\mathcal{F}_k = \left\{\{v_0, \dots, v_k\} \subset \eta_t : \bigcap_0^k B(v_i, \frac{\delta_t}{2}) \neq \emptyset\right\}$

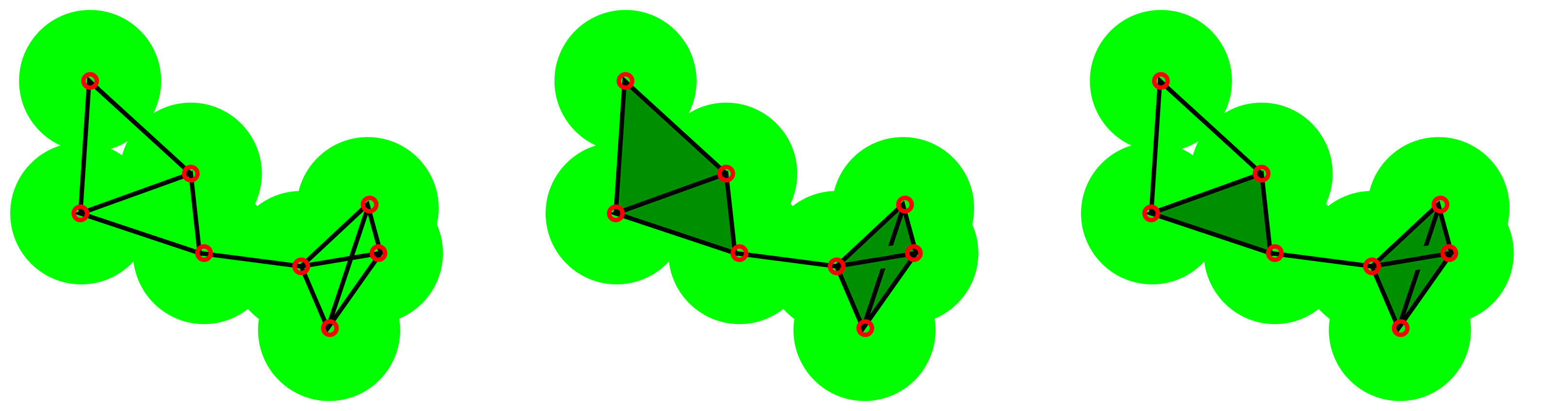

Fig. 1: The Gilbert graph, the Vietoris-Rips complex and the Čech complex of an 8 point set.

Geometry of Random Complexes

- **face numbers**, the $\boldsymbol{f}$-vector $(f_0, f_1 \dots)$ contains the number $f_i = |\mathcal{F}_i|$ of i-dimensional faces in a compact set:
  - expectation $\mathbb{E}\boldsymbol{f}$;
  - covariance matrix $\Sigma(\boldsymbol{f})$;
  - LLN, uni- and multivariate CLT's, limit theorems;
  - concentration inequalities.
- **percolation**, there exists a critical value v_{perc}:
  - for $t\delta_t < v_{\text{perc}}$ only finite components occur;
  - for $t\delta_t > v_{\text{perc}}$ there is a unique infinite component.
- **component counts**:
  - the expected number of components with at most k vertices;
  - CLT's and concentration inequalities.

Algebraic Properties of Random Complexes

- **(reduced) Euler characteristic** $\chi = \sum (-1)^i f_i$:
  - first two moments $\mathbb{E}\chi$, $\mathbb{V}\chi$, e.g., by discrete Morse theory;
  - deviation inequalities using Malliavin calculus.
- **(topological) Betti numbers** $\beta_k = \dim_K H_k(\Delta; K)$ for a field K and complex Δ:
  - in the sparse regime: $\mathbb{E}\beta_k$ and $\mathbb{V}\beta_k$, PLT and CLT;
  - in the thermodynamic regime: $\mathbb{E}\beta_k$ and $\mathbb{V}\beta_k$, SLN and CLT;
  - in the dense regime: order of $\mathbb{E}\beta_k$, threshhold for $\beta_k = 0$.

Metric Properties of Random Complexes

- **length power functional** $L^{(\tau)} = \frac{1}{2} \sum_{\mathcal{F}_1} \|x_1 - x_2\|^\tau$:
  - first two moments $\mathbb{E}L^{(\tau)}$, $\mathbb{V}L^{(\tau)}$, applied algebra, Mecke formula;
  - (multivariate) CLT's, deviation inequalities using Stein's method, Malliavin calculus.
- **volume power functional** $V_k^{(\tau)} = \frac{1}{k!} \sum_{\mathcal{F}_k} \lambda_k[x_1, \dots, x_k]^\tau$:
  - first two moments $\mathbb{E}V_k^{(\tau)}$, $\mathbb{V}V_k^{(\tau)}$, discrete geometry, Mecke formula;
  - (multivariate) CLT's using Stein's method, Malliavin calculus.

Aims

- **volume power functionals**: non-central limit theorems and concentration inequalities;
- **Euler characteristic**: concentration inequalities;
- $\boldsymbol{h}$**-vector**: $\boldsymbol{h} = (h_0, h_1 \dots)$ with $h_k = \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$; $h_{d-1} = (-1)^{d-1}\chi$; the probability that $\boldsymbol{h}$ is nonnegative, symmetric, or unimodal;
- **Cohen-Macaulay property:** the probability that random complexes are Cohen-Macaulay; implications on the $\boldsymbol{h}$-vector; yields many useful algebraic tools to investigate simplicial complexes and their properties;
- **algebraic Betti numbers** (generalize topological Betti numbers): expectation and variance of algebraic Betti numbers; proving deviations or concentration inequalities for Betti numbers of $\mathcal{V}(\eta_t, \delta_t)$ and $\mathcal{C}(\eta_t, \delta_t)$.

Working program

1. proving limit theorems and concentration results for $V_k^{(\tau)}$;
2. determining the top dimension of $\mathcal{V}(\eta_t, \delta_t)$ and $\mathcal{C}(\eta_t, \delta_t)$ via the $\boldsymbol{f}$-vector;
3. investigating the Euler characteristic by utilizing the knowledge on the top dimension;
4. $\boldsymbol{h}$-vector depends on $\boldsymbol{f}$-vector and dimension: investigating its asymptotic behaviour;
5. properties of links: determining the asymptotic distribution of their Betti numbers;
6. the Cohen-Macaulay property follows from properties of the Betti numbers of links;
7. comparism to results in the spherical and hyperbolic setting.

Project-related Work of the Applicants

[**Akinwande-Reitzner '20; Reitzner-Schulte-Thäle '17; Reitzner-Römer-Westenholz '23+**] limit behaviour of volume power functionals

[**Bachmann-Reitzner '18; Reitzner '13**] concentration inequalities for Poisson functionals, applied to the f-vector and general subgraph counts

[**Brun-Römer'08; Bruns-Koch-Römer'08; Bruns-Römer'07;Le-Römer'13**] study of objects of discrete mathematics with methods from combinatorial algebra

[**Bruns-Conca-Römer '11**] vanishing results for Betti numbers of Veronese algebras

[**Edelsbrunner-Nikitenko-Reitzner'17**] discrete Morse theory for Delaunay mosaics

[**Grygierek-Juhnke-K.-Reitzner-Römer-Röndigs '19**] complicated topological structure of $\mathcal{V}(\eta_t, \delta_t)$ and $\mathcal{C}(\eta_t, \delta_t)$

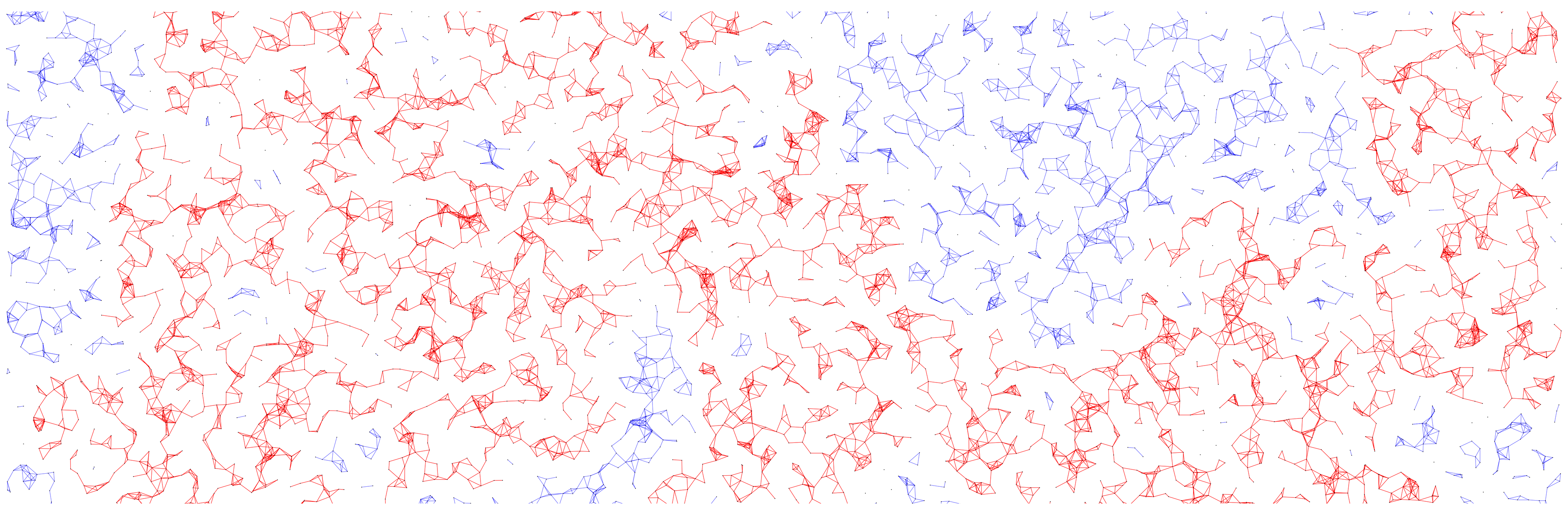

Fig. 2: A computer simulation of Percolation in the Gilbert graph in $d = 2$; infinite component in red; finite components in blue.